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Some New Weighted Hardy-Type Inequalities On Time Scales With Negative Exponents

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ABSTRACT

In this paper, we obtain some new Hardy-type inequalities with negative exponents on time scales. These inequalities, as a special case, when the time scale $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{N}$, contain some new integral and discrete inequalities with negative exponents. To the best of the authors' knowledge, the findings presented in this paper are assumed to be new in literature.

1. Introduction

In 1920, Hardy [1] established his well-known discrete inequality

$$\sum_{r=1}^{\infty}\left(\frac{1}{r}\sum_{n=1}^{r} v_n\right)^{l} \leqslant \left(\frac{l}{l-1}\right)^{l}\sum_{r=1}^{\infty} v_r^{l},\ l>1 \tag{1.1}$$

where $\{v_r\}_{r=1}^{\infty}$ is a nonnegative sequence such that $\sum_{r=1}^{\infty} v_r^{l}<\infty$. In [2], Hardy proved the integral analogue of (1.1) which demonstrated that for $l>1$ and $g(\xi)$ is a nonnegative function over any finite interval $(0,\xi)$ and $\int_0^{\infty} g^{l}(\xi)d\xi<\infty$,then

$$\int_0^{\infty}\left(\frac{1}{\xi}\int_0^{\xi} g(\tau)d\tau\right)^{l} d\xi \leqslant \left(\frac{l}{l-1}\right)^{l}\int_0^{\infty} g^{l}(\xi)d\xi. \tag{1.2}$$

Lots of generalizations and extensions of Hardy inequality with a positive parameter have been presented in the literature. For further details, the interested reader is referred to the papers [1, 3-8] and the books [9-12]. In the following, we briefly point out some of these extensions that support the results in this paper.

In 1925, Hardy [2] gave a generalization to (1.1) and showed that if $l>1,\{g_r\}_{r=1}^{\infty}$, $\{v_r\}_{r=1}^{\infty}$ are positive sequences, and $V_r=\sum_{n-1}^{r} v_n$, then

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$$\sum_{r=1}^{\infty} \frac{v_r}{V_r^l}\left(\sum_{n=1}^{r} v_n g(n)\right)^l \leqslant \left(\frac{l}{l-1}\right)^l \sum_{r=1}^{\infty} v_r g_r^l \tag{1.3}$$

The constants in (1.1), (1.2), and (1.3) are the best possible.
In 1928, Copson [5] introduced an extended version of (1.3) and proved the discrete inequality

$$\sum_{r=1}^{\infty} \frac{v_r}{V_r^c}\left(\sum_{n=1}^{r} v_n g_n\right)^l \leqslant \left(\frac{l}{c-1}\right)^l \sum_{r=1}^{\infty} v_r V_r^{l-c} g_r^l, \tag{1.4}$$

where $l \geq c > 1, \{g_r\}_{r=1}^{\infty}$ and $\{v_r\}_{r=1}^{\infty}$ are positive sequences and $V_r = \sum_{n=1}^{r} v_n$.
He also proved the discrete inequality.

$$\sum_{r=1}^{\infty} \frac{v_r}{V_r^c}\left(\sum_{n=r}^{\infty} v_n g_n\right)^l \leqslant \left(\frac{l}{1-c}\right)^l \sum_{r=1}^{\infty} v_r V_r^{l-c} g_r^l \tag{1.5}$$

where $0 \leqslant c < 1$ and $l > 1$.
In [8], Leindler derived the discrete inequality

$$\sum_{r=1}^{\infty} \frac{v_r}{(V_r^*)^c}\left(\sum_{n=1}^{r} v_n g_n\right)^l \leqslant \left(\frac{l}{1-c}\right)^l \sum_{r=1}^{\infty} v_r (V_r^*)^{l-c} g_r^l \tag{1.6}$$

where $0 \leqslant c < 1, l > 1, V_r^* = \sum_{n=r}^{\infty} v_n$ and $\sum_{n=r}^{\infty} v_n < \infty$. In [3], Bennett proved a dual of (1.6) by assuming that if $\sum_{n=r}^{\infty} v_n < \infty$ and $1 < c \leqslant l$, then

$$\sum_{r=1}^{\infty} \frac{v_r}{(V_r^*)^c}\left(\sum_{n=r}^{\infty} v_n g_n\right)^l \leqslant \left(\frac{l}{c-1}\right)^l \sum_{r=1}^{\infty} v_r (V_r^*)^{l-c} g_r^l \tag{1.7}$$

The constants in (1.4), (1.5), (1.6), and (1.7) are best possible.
In 1928, Hardy [13] extended (1.2) by proving that if $l, \gamma > 1$ and $g(\xi)$ is a nonnegative integrable function on $(0, \xi)$ such that $\int_0^{\infty} g^l(\xi) d\xi < \infty$, then

$$\int_0^{\infty} \frac{1}{\xi^{\gamma}}\left(\int_0^{\xi} g(\tau) d\tau\right)^l d\xi \leqslant \left(\frac{l}{\gamma-1}\right)^l \int_0^{\infty} \frac{1}{\xi^{\gamma-l}} g^l(\xi) d\xi \tag{1.8}$$

and for $0 < \gamma \leqslant 1$, he proved that

$$\int_0^{\infty} \frac{1}{\xi^{\gamma}}\left(\int_{\xi}^{\infty} g(\tau) d\tau\right)^l d\xi \leqslant \left(\frac{l}{1-\gamma}\right)^l \int_0^{\infty} \frac{1}{\xi^{\gamma-l}} g^l(\xi) d\xi \tag{1.9}$$

Recently, a lot of generalizations and extensions of Hardy inequality with negative parameters have appeared. In the following, we recall some of these integral inequalities with power $l < 0$.
In [14], it was demonstrated by the authors that if $l < 0, c > 1, g(\xi) > 0$ and $\int_0^{\infty} \xi^{-c}(\xi g(\xi))^l d\xi < \infty$, then

$$\int_0^{\infty} \xi^{-c}\left(\int_{\xi}^{\infty} g(\tau) d\tau\right)^l d\xi \leq \left(\frac{l}{1-c}\right)^l \int_0^{\infty} \xi^{-c}(\xi g(\xi))^l d\xi \tag{1.10}$$

holds.
In the same paper, it was demonstrated that if $l < 0, c < 1, g(\xi) > 0$ and $\int_0^{\infty} \xi^{-c}(\xi g(\xi))^l d\xi < \infty, \xi^{-c}(\xi g(\xi))^l d\xi < \infty$, then the inequality

$$\int_0^{\infty} \xi^{-c}\left(\int_0^{\xi} g(\tau) d\tau\right)^l d\xi \leq \left(\frac{l}{c-1}\right)^l \int_0^{\infty} \xi^{-c}(\xi g(\xi))^l d\xi \tag{1.11}$$

holds. The constants in (1.10) and (1.11) are best possible.
In [15], it was demonstrated that if $l < 0, c < 1$ and $v(\xi), g(\xi)$ are positive functions, then

$$\int_0^{b} v(\xi) V^{-c}(\xi) G^l(\xi) d\xi \leq \left(\frac{l}{c-1}\right)^l \int_0^{b} v(\xi) V^{l-c}(\xi) g^l(\xi) d\xi \tag{1.12}$$

holds, where

$$V(\xi)=\int_0^{\xi} v(\tau)d\tau,\ G(\xi)=\int_0^{\xi} v(\tau)g(\tau)d\tau$$

In the same paper, they also proved the inequality

$$\int_0^b v(\xi)V^{-c}(\xi)\bar{G}^l(\xi)d\xi \leq \left(\frac{l}{1-c}\right)^l \int_0^b v(\xi)g^l(\xi)V^{l-c}(\xi)d\xi$$

where, $c>1$ and $\bar{G}(\xi)=\int_{\xi}^{\infty} g(\tau)v(\tau)d\tau$.

In [15], the authors showed that if $l<0$ and $\alpha \neq l-1$, then

$$\int_0^b v(\xi)V^{\alpha-l}(\xi)G^l(\xi)d\xi \leq \left(\frac{l}{|l-1-\alpha|}\right)^l \int_0^b v(\xi)V^{\alpha}(\xi)g^l(\xi)d\xi \tag{1.13}$$

holds, where

$$V(\xi)=\int_0^{\xi} v(\tau)d\tau,\ G(\xi)=\begin{cases}\int_0^{\xi} g(\tau)v(\tau)d\tau & \text{for } \alpha>l-1\\ \int_{\xi}^{\infty} g(\tau)v(\tau)d\tau & \text{for } \alpha<l-1\end{cases}$$

In [16], the authors demonstrated that, if $0\leq s_1<s_2\leq\infty, p<l<0, 0<c<1$ and $v(\xi), g(\xi)$ are non-negative Lebesgue measurable functions, then

$$\int_{s_1}^{s_2} \frac{v(\xi)}{V^c(\xi)}\mathcal{Z}_{v,1}^l(\xi)d\xi \leq \left(\frac{l}{l+c-1}\right)^l \left(\int_{s_1}^{s_2} v(\xi)d\xi\right)^{1-\frac{l}{p}} \left(\int_{s_1}^{s_2} \frac{v(\xi)}{V^{\frac{cp}{l}}(\xi)} g^p(\xi)d\xi\right)^{\frac{l}{p}} \tag{1.14}$$

where,

$$V(\xi)=\int_0^{\xi} v(\tau)d\tau \text{ and } \mathcal{Z}_{v,1}(\xi)=\frac{1}{V(\xi)}\int_{s_1}^{\xi} v(\tau)g(\tau)d\tau$$

In the same paper, the authors also demonstrated that if $c\geq l$ and $p<l<0$, then

$$\int_{s_1}^{s_2} \frac{v(\xi)\left(\mathcal{Z}_{v,2}\right)^l(\xi)}{V^{l-c}(\xi)}d\xi \leq \left(\frac{l}{l-c-1}\right)^l \left(\int_{s_1}^{s_2} v(\xi)d\xi\right)^{1-\frac{l}{p}} \left(\int_{s_1}^{s_2} \frac{v(\xi)g^p(\xi)}{V^{p-\frac{\gamma p}{l}}(\xi)}d\xi\right)^{\frac{l}{p}} \tag{1.15}$$

where,

$$\mathcal{Z}_{v,2}(\xi)=\int_{s_1}^{\xi} \frac{v(\tau)g(\tau)}{V(\tau)}d\tau$$

Recently, there is a significant attention on studying dynamic inequalities and their applications on time scales. These inequalities can be extended to the more general framework of time scales, which unifies discrete and continuous analysis into a single theory. For a more comprehensive understanding of these inequalities, see the books [17,18] and the papers $[19-28]$.

For completeness, we present some dynamic inequalities that motivate the material discussed in this paper.

The dynamic version of (1.2) was proved in [29]. The author demonstrated that if $l>1, g\in C_{rd}([a,\infty),\mathbb{R}^+)$and $\int_0^{\infty} g^l(\xi)\Delta\xi<\infty$, then

$$\int_a^{\infty}\left(\frac{1}{\sigma(\xi)-a}\int_a^{\sigma(\xi)} g(\tau)\Delta\tau\right)^l\Delta\xi \leqslant \left(\frac{l}{l-1}\right)^l \int_a^{\infty} g^l(\xi)\Delta\xi \tag{1.16}$$

If in addition $\frac{\mu(\xi)}{\xi}\to 0$ as $\xi\to\infty$, then $\left(\frac{l}{l-1}\right)^l$ is sharp. In [25], the authors derived the time scale version of (1.8) as

$$\int_a^{\infty}\frac{1}{\xi^{\gamma}}\left(\int_a^{\sigma(\xi)} g(\tau)\Delta\tau\right)^l\Delta\xi \leqslant \left(\frac{lM^{\gamma}}{\gamma-1}\right)^l \int_a^{\infty}\frac{1}{\xi^{\gamma-l}} g^l(\xi)\Delta\xi$$

where $l,\gamma > 1$ and $M > 0$ such that $\frac{\xi}{\sigma(\xi)} \geqslant \frac{1}{M}$. They also showed that if $\gamma < 1$, then

$$\int_a^\infty \frac{1}{\sigma^\gamma(\xi)}\left(\int_\xi^\infty g(\tau)\Delta\tau\right)^l \Delta\xi \leqslant \left(\frac{l}{1-\gamma}\right)^l \int_a^\infty \frac{1}{\sigma^{\gamma-l}(\xi)} g^l(\xi)\Delta\xi$$

In [24], the authors extended (1.4) and (1.5) to their time scale versions. In particular, they showed that if $l,\gamma > 1$, then

$$\int_a^\infty \frac{v(\xi)}{\left(V^\sigma(\xi)\right)^\gamma}\left(G^\sigma(\xi)\right)^l \Delta\xi \leqslant \left(\frac{l}{\gamma-1}\right)^l \int_a^\infty \frac{\left(V^\sigma(\xi)\right)^{\gamma(l-1)}}{(V(\xi))^{(\gamma-1)l}} v(\xi) g^l(\xi)\Delta\xi$$

where

$$V(\xi) = \int_a^\xi v(\tau)\Delta\tau \text{ and } G(\xi) = \int_a^\xi v(\tau)g(\tau)\Delta\tau, \quad \text{for any } \xi \in [a,\infty)_{\mathbb{T}}$$

and if $0 \leqslant \gamma < 1$, then

$$\int_a^\infty \frac{v(\xi)}{\left(V^\sigma(\xi)\right)^\gamma}(\bar{G}(\xi))^l \Delta\xi \leqslant \left(\frac{l}{1-\gamma}\right)^l \int_a^\infty v(\xi) g^l(\xi)\left(V^\sigma(\xi)\right)^{l-\gamma}\Delta\xi \tag{1.17}$$

where

$$\bar{G}(\xi) = \int_\xi^\infty v(\tau)g(\tau)\Delta\tau, \text{ for any } \xi \in [a,\infty)_{\mathbb{T}}$$

In [30], the authors extended (1.6) and (1.7) to their time scale versions. In particular, they showed that if $p > 1$ and $0 \leqslant \gamma < 1$, then

$$\int_a^\infty \frac{v(\xi)}{\Lambda^\gamma(\xi)}\left(G^\sigma(\xi)\right)^p \Delta\xi \leqslant \left(\frac{p}{1-\gamma}\right)^p \int_a^\infty \frac{v(\xi)}{\Lambda^{\gamma-p}(\xi)} g^p(\xi)\Delta\xi \tag{1.18}$$

where

$$\Lambda(\xi) = \int_\xi^\infty v(\tau)\Delta\tau$$

and if $p \geqslant \gamma > 1$, then

$$\int_a^\infty \frac{v(\xi)}{\Lambda^\gamma(\xi)}(\bar{G}(\xi))^p \Delta\xi \leqslant \left(\frac{p}{\gamma-1}\right)^p \int_a^\infty \frac{v(\xi)}{\Lambda^{\gamma-p}(\xi)} g^p(\xi)\Delta\xi \tag{1.19}$$

It is important to consider that all the previously mentioned dynamic inequalities discussed the case of positive exponents. The question is whether it is possible to establish similar dynamic inequalities of negative exponents.

In this paper, we will provide an affirmative answer to the previous question by proving some new dynamic inequalities with negative power and getting the integral and discrete analogies of these inequalities. The structure of the paper is as follows: In the next section, we review some definitions related to time-scale calculus. In Section 3, we state and prove our results.

2. Preliminaries

The time-scale calculus was initiated to create a theory that unifies difference and differential equations theories. Understanding its basics and applications can be found in Bohner and Peterson's books [31- 35]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. For $\tau \in \mathbb{T}$, the definition of the forward and backward jump operators $\sigma, \rho: \mathbb{T} \to \mathbb{T}$ is $\sigma(\tau) := \inf\{z \in \mathbb{T}: z > \tau\}$ and $\rho(\tau) := \sup\{z \in \mathbb{T}: z < \tau\}$. A function $g: \mathbb{T} \to \mathbb{R}$ is called right dense-continuous (rd-continuous) if it is continuous at all right-dense points in $\mathbb{T}$ and there exists a finite left limit at all left-dense points in $\mathbb{T}$. We use $C_{rd}(\mathbb{T})$, to represent the set of all rd-continuous functions. We have $g^\sigma(\tau) := g(\sigma(\tau))$ and define $[s_1, s_2]_{\mathbb{T}}$ by $[s_1, s_2]_{\mathbb{T}} = \{\tau \in \mathbb{T}: s_1 \leqslant \tau \leqslant s_2\}$. For a function $g: \mathbb{T} \to \mathbb{R}, g^\Delta(\tau)$ is defined as the number that satisfies the following condition: for $\tau \in \mathbb{T}$ and $\epsilon > 0$, there exists a neighborhood U of τ such that

$$\left|[g^\sigma(\tau) - g(s)] - g^\Delta(\tau)[\sigma(\tau) - s]\right| \leqslant \epsilon|\sigma(\tau) - s|, \quad \text{for } s \in U$$

In this case, $g^{\Delta}(\tau)$ is said to be the delta derivative of g at τ. If $\Psi^{\Delta}(\tau)=\psi(\tau)$, then $\Psi:\mathbb{T}\to\mathbb{R}$ is referred to as the delta antiderivative of ψ. The integral of ψ is given by

$$\int_{\tau_0}^{\tau}\psi(\xi)\Delta\xi=\Psi(\tau)-\Psi(\tau_0),\text{ for }\tau_0,\tau\in\mathbb{T}$$

One of the most widely used rules in time scale calculus is the chain rule, which states:

$$(\phi\circ\psi)^{\Delta}(\xi)=\int_0^1\phi'[\psi(\xi)+h\mu(\xi)\psi^{\Delta}(\xi)]dh\psi^{\Delta}(\xi)\tag{2.1}$$

where, $\psi:\mathbb{T}\to\mathbb{R}$ is delta differentiable and $\phi:\mathbb{R}\to\mathbb{R}$ is continuously differentiable (see [31, Theorem 1.90]). A special case of (2.1) is

$$\left(\phi^{\gamma}(\xi)\right)^{\Delta}=\gamma\phi^{\Delta}(\xi)\int_0^1(h\phi^{\sigma}+(1-h)\phi)^{\gamma-1}dh,\ \gamma\in\mathbb{R}\tag{2.2}$$

Lemma 2.1 (Integration by Parts [17]). If $s_1,s_2\in\mathbb{T}$ and $w(\xi),u(\xi)\in C_{rd}([s_1,s_2]_{\mathbb{T}},\mathbb{R}^{+})$, then

$$\int_{s_1}^{s_2}w^{\sigma}(\xi)u^{\Delta}(\xi)\Delta\xi=[w(\xi)u(\xi)]_{s_1}^{s_2}-\int_{s_1}^{s_2}w^{\Delta}(\xi)u(\xi)\Delta\xi\tag{2.3}$$

and

$$\int_{s_1}^{s_2}w(\xi)u^{\Delta}(\xi)\Delta\xi=[w(\xi)u(\xi)]_{s_1}^{s_2}-\int_{s_1}^{s_2}w^{\Delta}(\xi)u^{\sigma}(\xi)\Delta\xi\tag{2.4}$$

Lemma 2.2 (Weighted Hölder's Inequality [17]). If $s_1,s_2\in\mathbb{T}$ and $v(\xi),u(\xi),w(\xi)\in C_{rd}([s_1,s_2]_{\mathbb{T}},\mathbb{R}^{+})$, then

$$\int_{s_1}^{s_2}v(\xi)u(\xi)w(\xi)\Delta\xi\le\left[\int_{s_1}^{s_2}v(\xi)u^{\alpha}(\xi)\Delta\xi\right]^{\frac{1}{\alpha}}\left[\int_{s_1}^{s_2}v(\xi)w^{\alpha'}(\xi)\Delta\xi\right]^{\frac{1}{\alpha'}}\tag{2.5}$$

where $\alpha>1$, and $\frac{1}{\alpha'}+\frac{1}{\alpha}=1$.

Lemma 2.3 (Reversed Weighted Hölder's Inequality [17]). If $s_1,s_2\in\mathbb{T}$ and $v(\xi),u(\xi),w(\xi)\in C_{rd}([s_1,s_2]_{\mathbb{T}},\mathbb{R}^{+})$, the

$$\int_{s_1}^{s_2}v(\xi)u(\xi)w(\xi)\Delta\xi\ge\left[\int_{s_1}^{s_2}v(\xi)u^{\alpha}(\xi)\Delta\xi\right]^{\frac{1}{\alpha}}\left[\int_{s_1}^{s_2}v(\xi)w^{\alpha'}(\xi)\Delta\xi\right]^{\frac{1}{\alpha'}}\tag{2.6}$$

where $\alpha<0$ or $0<\alpha<1$, and $\frac{1}{\alpha'}+\frac{1}{\alpha}=1$.

3. Main Results

Throughout the rest of the paper, we will assume that the functions are positive and that all integrals involved are presumed to exist. We also assume that $v(\xi)$ and $z(\xi)\in C_{rd}[[0,\infty)_{\mathbb{T}},\mathbb{R}^{+}]$. We can now present and prove our results.

Theorem 3.1. If $a,b\in\mathbb{T},0<a<b$ and $l<0$, then

$$\int_a^b v(\xi)\left(\Omega^{\sigma}(\xi)\right)^{l}\Delta\xi\le(-l)^{l}\int_a^b v^{1-l}(\xi)z^{l}(\xi)\mathcal{V}^{l}(\xi)\Delta\xi\tag{3.1}$$

where,

$$\mathcal{V}(\xi)=\int_a^{\xi}v(\tau)\Delta\tau\ \text{ and }\ \Omega(\xi)=\int_0^{\xi}z(\tau)\Delta\tau$$

Proof. By employing (2.3) to the left-hand side of (3.1) with $u^{\Delta}(\xi)=v(\xi)$ and $w^{\sigma}(\xi)=\left(\Omega^{\sigma}(\xi)\right)^{l}$, we obtain

$$\int_a^b v(\xi)\left(\Omega^{\sigma}(\xi)\right)^{l}\Delta\xi=u(\xi)\Omega^{l}(\xi)|_a^b+\int_a^b u(\xi)\left(-\left(\Omega^{l}(\xi)\right)^{\Delta}\right)\Delta\xi\tag{3.2}$$

where

$$u(\xi)=\int_a^{\xi}v(\tau)\Delta\tau=\mathcal{V}(\xi)$$

Since $u(a)=0$, we conclude that

$$\int_a^b v(\xi)\left(\Omega^\sigma(\xi)\right)^l \Delta\xi = u(b)\Omega^l(b) - \int_a^b \mathcal{V}(\xi)\left(\Omega^l(\xi)\right)^\Delta \Delta\xi$$

By employing (2.2) to the term $\left(\Omega^l(\xi)\right)^\Delta$ and taking into account that $\Omega^\Delta(\xi) = z(\xi) > 0$, we obtain

$$\begin{aligned}\left(\Omega^l(\xi)\right)^\Delta &= lz(\xi)\int_0^1 \left(h\Omega^\sigma(\xi) + (1-h)\Omega(\xi)\right)^{l-1} dh \\ &\le lz(\xi)\left(\Omega^\sigma(\xi)\right)^{l-1}\end{aligned}$$

Thus

$$-\left(\Omega^l(\xi)\right)^\Delta \ge -lz(\xi)\left(\Omega^\sigma(\xi)\right)^{l-1} \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\int_a^b v(\xi)\left(\Omega^\sigma(\xi)\right)^l \Delta\xi \ge -l\int_a^b z(\xi)\mathcal{V}(\xi)\left(\Omega^\sigma(\xi)\right)^{l-1}\Delta\xi$$

By employing (2.6) on $\int_a^b v^{\frac{l-1}{l}}(\xi)v^{\frac{1-l}{l}}(\xi)z(\xi)\mathcal{V}(\xi)\left(\Omega^\sigma(\xi)\right)^{l-1}\Delta\xi$, with indices $\frac{l}{l-1}$ and l, we get

$$\int_a^b v^{\frac{l-1}{l}}(\xi)v^{\frac{1-l}{l}}(\xi)z(\xi)\mathcal{V}(\xi)\left(\Omega^\sigma(\xi)\right)^{l-1}\Delta\xi$$

Proof. Substituting from (3.6) into (3.5), we get

$$\int_a^b v(\xi)\left(\Omega^\sigma(\xi)\right)^l \Delta\xi \ge -l\left(\int_a^b v(\xi)\left(\Omega^\sigma(\xi)\right)^l\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_a^b v^{1-l}(\xi)z^l(\xi)\mathcal{V}^l(\xi)\Delta\xi\right)^{\frac{1}{l}}$$

Thus

$$\left(\int_a^b v(\xi)\left(\Omega^\sigma(\xi)\right)^l\Delta\xi\right)^{\frac{1}{l}} \ge -l\left(\int_a^b v^{1-l}(\xi)z^l(\xi)\mathcal{V}^l(\xi)\Delta\xi\right)^{\frac{1}{l}}$$

Therefore,

$$\int_a^b v(\xi)\left(\Omega^\sigma(\xi)\right)^l\Delta\xi \le (-l)^l\int_a^b v^{1-l}(\xi)z^l(\xi)\mathcal{V}^l(\xi)\Delta\xi$$

which is (3.1).

Remark 3.1. It is worth mentioning that the inequality (3.1) represents a corresponding formula, with a negative exponent, to the dynamic inequality proved by Saker in [36, Theorem 2.3].

Remark 3.2. If $\mathbb{T} = \mathbb{R}$ in (3.1), then

$$\int_a^b v(\xi)(\Omega(\xi))^l d\xi \le (-l)^l\int_a^b v^{1-l}(\xi)z^l(\xi)\mathcal{V}^l(\xi)d\xi$$

where,

$$\mathcal{V}(\xi) = \int_a^\xi v(\tau)d\tau \ \text{ and } \ \Omega(\xi) = \int_0^\xi z(\tau)d\tau$$

Remark 3.3. If $\mathbb{T} = \mathbb{N}, b = \infty, \{z_r\}_{r=1}^\infty$ is positive sequence and $\sum_{r=1}^\infty v_r^{1-l}z_r^l\mathcal{V}_r^l$ is convergent sequence in (3.1), then

$$\sum_{r=1}^\infty v_r\left(\sum_{n=0}^r z_n\right)^l \le (-l)^l\sum_{r=1}^\infty v_r^{1-l}z_r^l\mathcal{V}_r^l$$

where

$$\mathcal{V}_r = \sum_{n=1}^{r-1} v_n$$

Theorem 3.2. Assume $a, b \in \mathbb{T}, 0 < a < b$ and $-\infty < p \le l < 0$. If $0 \le c < 1$, then

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi \le \left(\frac{l}{c-1}\right)^l\int_a^b \frac{v(\xi)z^l(\xi)}{\mathcal{V}^{c-l}(\xi)}\Delta\xi \tag{3.7}$$

and

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi \le \left(\frac{l}{c-1}\right)^l\left(\int_a^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}}\left(\int_a^b v(\xi)\mathcal{V}^{p\left(1-\frac{c}{l}\right)}(\xi)z^p(\xi)\Delta\xi\right)^{\frac{l}{p}} \tag{3.8}$$

where,

$$\mathcal{V}(\xi) = \int_a^\xi v(\tau)\Delta\tau \text{ and } \mathcal{Z}(\xi) = \int_0^\xi v(\tau)z(\tau)\Delta\tau \tag{3.9}$$

Proof. By employing (2.3) to the left-hand side of (3.7) with $u^\Delta(\xi) = \frac{v(\xi)}{\mathcal{V}^c(\xi)}$ and $w^\sigma(\xi) = \left(\mathcal{Z}^\sigma(\xi)\right)^l$, we obtain

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi = u(\xi)\mathcal{Z}^l(\xi)|_a^b + \int_a^b u(\xi)\left(-\left(\mathcal{Z}^l(\xi)\right)^\Delta\right)\Delta\xi \tag{3.10}$$

where

$$u(\xi) = \int_a^\xi \frac{v(\tau)}{\mathcal{V}^c(\tau)}\Delta\tau$$

Since $u(a) = 0$, we conclude that

$$\begin{aligned}\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi &= u(b)\mathcal{Z}^l(b) - \int_a^b u(\xi)\left(\mathcal{Z}^l(\xi)\right)^\Delta\Delta\xi \\ &\ge -\int_a^b u(\xi)\left(\mathcal{Z}^l(\xi)\right)^\Delta\Delta\xi\end{aligned}$$

By employing (2.2) and taking into account that $\mathcal{V}^\Delta(\xi) = v(\xi) > 0$ and $0 \le c < 1$, we obtain

$$\begin{aligned}\left(\mathcal{V}^{1-c}(\xi)\right)^\Delta &= (1-c)\mathcal{V}^\Delta(\xi)\int_0^1 \left(h\mathcal{V}^\sigma(\xi) + (1-h)\mathcal{V}(\xi)\right)^{-c}dh \\ &\le (1-c)v(\xi)\int_0^1 (h\mathcal{V}(\xi) + (1-h)\mathcal{V}(\xi))^{-c}dh \\ &= (1-c)v(\xi)\mathcal{V}^{-c}(\xi)\end{aligned}$$

Thus

$$v(\xi)\mathcal{V}^{-c}(\xi) \ge \frac{1}{1-c}\left(\mathcal{V}^{1-c}(\xi)\right)^\Delta$$

and then, we have

$$u(\xi) = \int_a^\xi \frac{v(\tau)}{\mathcal{V}^c(\tau)}\Delta\tau \ge \frac{1}{1-c}\int_a^\xi \left(\mathcal{V}^{1-c}(\tau)\right)^\Delta\Delta\tau = \frac{1}{1-c}\mathcal{V}^{1-c}(\xi) \tag{3.11}$$

By employing (2.2) again to the term $\left(\mathcal{Z}^l(\xi)\right)^\Delta$, we have

$$\begin{aligned}\left(\mathcal{Z}^l(\xi)\right)^\Delta &= \left(\left(\int_0^\xi v(\tau)z(\tau)\Delta\tau\right)^l\right)^\Delta \\ &= lv(\xi)z(\xi)\int_0^1 \left(h\left(\int_0^{\sigma(\xi)} v(\tau)z(\tau)\Delta\tau\right) + (1-h)\left(\int_0^\xi v(\tau)z(\tau)\Delta\tau\right)\right)^{l-1} dh \\ &\le lv(\xi)z(\xi)\left(\int_0^{\sigma(\xi)} v(\tau)z(\tau)\Delta\tau\right)^{l-1}\end{aligned}$$

Thus

$$-\left(\mathcal{Z}^l(\xi)\right)^\Delta \ge -lv(\xi)z(\xi)\left(\mathcal{Z}^\sigma(\xi)\right)^{l-1} \tag{3.12}$$

From (3.11) and (3.12), we get

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi \ge \frac{-l}{1-c}\int_a^b \frac{v(\xi)z(\xi)\left(\mathcal{Z}^\sigma(\xi)\right)^{l-1}}{\mathcal{V}^{c-1}(\xi)}\Delta\xi \tag{3.13}$$

By employing (2.6) on $\int_a^b \frac{v(\xi)z(\xi)\left(\mathcal{Z}^\sigma(\xi)\right)^{l-1}}{\mathcal{V}^{c-1}(\xi)}\Delta\xi$, with indices $\frac{l}{l-1}$ and l, we get

$$\int_a^b \frac{v(\xi)z(\xi)\left(\mathcal{Z}^\sigma(\xi)\right)^{l-1}}{\mathcal{V}^{c-1}(\xi)}\Delta\xi \geq \left(\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_a^b \frac{v(\xi)z^l(\xi)}{\mathcal{V}^{c-l}(\xi)}\Delta\xi\right)^{\frac{1}{l}} \tag{3.14}$$

Substituting from (3.14) into (3.13), we get

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi \geq \frac{l}{c-1}\left(\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_a^b \frac{v(\xi)z^l(\xi)}{\mathcal{V}^{c-l}(\xi)}\Delta\xi\right)^{\frac{1}{l}}$$

thus

$$\left(\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi\right)^{\frac{1}{l}} \geq \frac{l}{c-1}\left(\int_a^b \frac{v(\xi)z^l(\xi)}{\mathcal{V}^{c-l}(\xi)}\Delta\xi\right)^{\frac{1}{l}}$$

Therefore,

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi \leq \left(\frac{l}{c-1}\right)^l\int_a^b \frac{v(\xi)z^l(\xi)}{\mathcal{V}^{c-l}(\xi)}\Delta\xi$$

which is (3.7).

By employing (2.5), with conjugate exponents $\frac{p}{p-l}$ and $\frac{p}{l}$, on $\int_a^b \frac{v(\xi)z^l(\xi)}{\mathcal{V}^{c-l}(\xi)}\Delta\xi$, we get

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi \leq \left(\frac{l}{c-1}\right)^l\left(\int_a^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}}\left(\int_a^b v(\xi)\mathcal{V}^{p\left(1-\frac{c}{l}\right)}(\xi)z^p(\xi)\Delta\xi\right)^{\frac{l}{p}}$$

which provides (3.8).

Corollary 3.1. If $v(\tau)=1$ in Theorem 3.2, then $\mathcal{V}(\xi)=\xi-a$ and we deduce under the assumption $b=\infty$ that, the following inequality

$$\int_a^\infty (\xi-a)^{-c}\left(\int_0^{\sigma(\xi)} z(\tau)\Delta\tau\right)^l\Delta\xi \leq \left(\frac{l}{c-1}\right)^l\int_a^\infty (\xi-a)^{l-c}z^l(\xi)\Delta\xi \tag{3.15}$$

holds.

Remark 3.4. If $\mathbb{T}=\mathbb{N}, b=\infty$ and $\{z_r\}_{r=1}^\infty, \{v_r\}_{r=1}^\infty$ are positive sequences such that $\sum_{r=1}^\infty v_r\mathcal{V}_r^{l-c}z_r^l$ is convergent sequence and $\mathcal{V}_r=\sum_{n=1}^{r-1}v_n$ in (3.7), then

$$\sum_{r=1}^\infty \frac{v_r}{\mathcal{V}_r^c}\left(\sum_{n=0}^r v_nz_n\right)^l \leq \left(\frac{l}{c-1}\right)^l\sum_{r=1}^\infty v_rz_r^l\mathcal{V}_r^{l-c}$$

which is similar to (1.4) but with a negative exponent l and $0\leq c<1$.

Remark 3.5. If $\mathbb{T}=\mathbb{R}$ and $b=\infty$ in (3.7), then we have the following Coposon-type inequality with a negative parameter

$$\int_a^\infty \frac{v(\xi)}{\mathcal{V}^c(\xi)}\left(\int_0^\xi v(\tau)z(\tau)d\tau\right)^l d\xi \leq \left(\frac{l}{c-1}\right)^l\int_a^\infty \frac{v(\xi)z^l(\xi)}{\mathcal{V}^{c-l}(\xi)}d\xi$$

Theorem 3.3. Assume $b\in\mathbb{T}, b<\infty$ and $-\infty<p\leq l<0$. If $0\leq c<1$, then

$$\int_0^b \frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{\mathcal{Z}}^l(\xi)\Delta\xi \leq \left(\frac{l}{c-1}\right)^l\int_0^b \frac{v(\xi)z^l(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{c-l}}\Delta\xi \tag{3.16}$$

and

$$\int_a^b \frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{\mathcal{Z}}^l(\xi)\Delta\xi \leq \left(\frac{l}{c-1}\right)^l\left(\int_a^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}}\left(\int_a^b v(\xi)\left(\Lambda^\sigma(\xi)\right)^{p\left(1-\frac{c}{l}\right)}z^p(\xi)\Delta\xi\right)^{\frac{l}{p}},$$

where,

$$\Lambda(\xi)=\int_\xi^b v(\tau)\Delta\tau \text{ and } \overline{\mathcal{Z}}(\xi)=\int_\xi^\infty v(\tau)z(\tau)\Delta\tau \tag{3.17}$$

Proof. By employing (2.4) to the left-hand side of (3.16) with $u^\Delta(\xi)=\frac{v(\xi)}{\Lambda^c(\xi)}$ and $w(\xi)=\overline{\mathcal{Z}}^l(\xi)$, we obtain

$$\int_0^b \frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{Z}^l(\xi)\Delta\xi = u(\xi)\overline{Z}^l(\xi)\Big|_0^b + \int_0^b \left(-u^\sigma(\xi)\right)\left(\overline{Z}^l(\xi)\right)^\Delta \Delta\xi \tag{3.18}$$

where

$$u(\xi) = -\int_\xi^b \frac{v(\tau)}{\left(\Lambda^\sigma(\tau)\right)^c}\Delta\tau$$

Since $u(b) = 0$ in (3.18), we conclude that

$$\int_0^b \frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{Z}^l(\xi)\Delta\xi = -u(0)\overline{Z}^l(0) - \int_0^b u^\sigma(\xi)\left(\overline{Z}^l(\xi)\right)^\Delta \Delta\xi$$

$$\geq -\int_0^b u^\sigma(\xi)\left(\overline{Z}^l(\xi)\right)^\Delta \Delta\xi$$

By employing (2.2) and taking into account that $\Lambda^\Delta(\xi) = -v(\xi) < 0$ and $0 \leq c < 1$, we obtain

$$\left(\Lambda^{1-c}(\xi)\right)^\Delta = (1-c)\Lambda^\Delta(\xi)\int_0^1 \left(h\Lambda^\sigma(\xi) + (1-h)\Lambda(\xi)\right)^{-c} dh$$

$$\geq (c-1)v(\xi)\int_0^1 \left(h\Lambda^\sigma(\xi) + (1-h)\Lambda^\sigma(\xi)\right)^{-c} dh$$

$$= (c-1)v(\xi)\left(\Lambda^\sigma(\xi)\right)^{-c}$$

Thus

$$v(\xi)\left(\Lambda^\sigma(\xi)\right)^{-c} \geq \frac{1}{c-1}\left(\Lambda^{1-c}(\xi)\right)^\Delta$$

and then, we have

$$-u^\sigma(\xi) = \int_{\sigma(\xi)}^b \frac{v(\tau)}{\left(\Lambda^\sigma(\tau)\right)^c}\Delta\tau \geq \frac{1}{c-1}\int_{\sigma(\xi)}^b \left(\Lambda^{1-c}(\tau)\right)^\Delta \Delta\tau \geq \frac{1}{1-c}\left(\Lambda^\sigma(\xi)\right)^{1-c} \tag{3.19}$$

By employing (2.2) again to the term $\left(\overline{Z}^l(\xi)\right)^\Delta$, we have

$$\left(\overline{Z}^l(\xi)\right)^\Delta = \left(\left(\int_\xi^\infty v(\tau)z(\tau)\Delta\tau\right)^l\right)^\Delta$$

$$= -lv(\xi)z(\xi)\int_0^1 \left(h\left(\int_{\sigma(\xi)}^\infty v(\tau)z(\tau)\Delta\tau\right) + (1-h)\left(\int_\xi^\infty v(\tau)z(\tau)\Delta\tau\right)\right)^{l-1} dh$$

From (3.19) and (3.20), we get

$$\int_0^b \frac{v(\tau)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{Z}^l(\xi)\Delta\xi \geq \frac{-l}{1-c}\int_0^b \frac{v(\xi)z(\xi)\overline{Z}^{l-1}(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{c-1}}\Delta\xi \tag{3.21}$$

By employing (2.6) on $\int_0^b \frac{v(\xi)z(\xi)\overline{Z}^{l-1}(\xi)}{(\Lambda^\sigma(\xi))^{c-1}}\Delta\xi$, with indices $\frac{l}{l-1}$ and l, we get

$$\int_0^b \frac{v(\xi)z(\xi)\overline{Z}^{l-1}(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{c-1}}\Delta\xi \geq \left(\int_0^b \frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{Z}^l(\xi)\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_0^b \frac{v(\xi)z^l(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{c-l}}\Delta\xi\right)^{\frac{1}{l}}. \tag{3.22}$$

Substituting from (3.22) into (3.21), we get

$$\int_0^b \frac{v(\tau)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{Z}^l(\xi)\Delta\xi \geq \frac{-l}{1-c}\left(\int_0^b \frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{Z}^l(\xi)\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_0^b \frac{v(\xi)z^l(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{c-l}}\Delta\xi\right)^{\frac{1}{l}}$$

Thus

$$\left(\int_0^b \frac{v(\tau)}{\left(\Lambda^\sigma(\xi)\right)^c}\overline{z}^l(\xi)\Delta\xi\right)^{\frac{1}{l}} \geq \frac{l}{c-1}\left(\int_0^b \frac{v(\xi)z^l(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{c-l}}\Delta\xi\right)^{\frac{1}{l}}$$

Therefore,

$$\int_0^b \frac{v(\tau)}{\left(\Lambda^\sigma(\xi)\right)^2}\overline{\mathcal{Z}}^l(\xi)\Delta\xi \leq \left(\frac{l}{c-1}\right)^l \int_0^b \frac{v(\xi)z^l(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{c-l}}\Delta\xi,$$

which is (3.16).

By employing (2.5), with conjugate exponents $\frac{p}{p-l}$ and $\frac{p}{l}$, on $\int_0^b \frac{v(\xi)z^l(\xi)}{(\Lambda^\sigma(\xi))^{c-l}}\Delta\xi$, we get

$$\int_0^b \frac{v(\xi)}{\Lambda^c(\xi)}\overline{\mathcal{Z}}^l(\xi)\Delta\xi \leq \left(\frac{l}{c-1}\right)^l \left(\int_0^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}} \left(\int_0^b v(\xi)\left(\Lambda^\sigma(\xi)\right)^{p\left(1-\frac{c}{l}\right)} z^p(\xi)\Delta\xi\right)^{\frac{l}{p}}$$

which provides (3.17).

Remark 3.6. It is worth mentioning that inequality (3.16) represents a corresponding formula, with a negative exponent, to the inequality (1.18).

Remark 3.7. If $\mathbb{T}=\mathbb{R}$ and $b<\infty$ in (3.16), then

$$\int_0^b \frac{v(\xi)}{\Lambda^c(\xi)}\left(\int_\xi^\infty v(\tau)z(\tau)d\tau\right)^l d\xi \leq \left(\frac{l}{c-1}\right)^l \int_0^b \frac{v(\xi)z^l(\xi)}{\Lambda^{c-l}(\xi)}d\xi \tag{3.23}$$

Remark 3.8. If $\mathbb{T}=\mathbb{N}$ and $\{z_r\}_{r=1}^\infty, \{v_r\}_{r=1}^\infty$ are positive sequences such that $\sum_{r=1}^{b-1} v_r z_r^l \Lambda_r^{l-c}$ is convergent sequence and $\Lambda_r=\sum_{n=r}^b v_n$ in (3.16), then

$$\sum_{r=1}^{b-1}\frac{v_r}{\Lambda_r^c}\left(\sum_{n=r}^\infty v_n z_n\right)^l \leq \left(\frac{l}{c-1}\right)^l \sum_{r=1}^{b-1} v_r \Lambda_r^{l-c} z_r^l$$

which is a corresponding formula to (1.6), but with negative exponent l.

Theorem 3.4. Assume $a,b\in\mathbb{T}, a>0$ and $-\infty<p\leq l<0$. If $0\leq c+l<1$, then

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi \leq \left(\frac{l}{c+l-1}\right)^l \int_a^b \frac{v(\xi)z^l(\xi)}{\mathcal{V}^c(\xi)}\Delta\xi \tag{3.24}$$

and

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi \leq \left(\frac{l}{c+l-1}\right)^l \left(\int_a^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}} \left(\int_a^b \frac{v(\xi)}{\mathcal{V}^{\frac{cP}{l}}(\xi)} z^p(\xi)\Delta\xi\right)^{\frac{l}{p}} \tag{3.25}$$

where $\mathcal{V}(\xi)$ and $\mathcal{Z}(\xi)$ are given in (3.9).

Proof. By employing (2.3) to the left-hand side of (3.24) with

$$u^\Delta(\xi)=\frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)\ \text{and}\ w^\sigma(\xi)}=\left(\int_0^{\sigma(\xi)} v(\tau)z(\tau)\Delta\tau\right)^l$$

we get

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi = u(\xi)w(\xi)|_a^b - \int_a^b u(\xi)w^\Delta(\xi)\Delta\xi$$

where

$$u(\xi)=\int_a^\xi \frac{v(\tau)}{\mathcal{V}^{cl}(\tau)}\Delta\tau$$

Since $u(a)=0$, we conclude that

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^\sigma(\xi)\right)^l\Delta\xi = u(b)\mathcal{Z}^l(b) - \int_a^b u(\xi)\left(\mathcal{Z}^l(\xi)\right)^\Delta\Delta\xi$$

$$\geq -\int_a^b u(\xi)\left(\mathcal{Z}^l(\xi)\right)^\Delta\Delta\xi.$$

From (2.2), we see that

$$\left(\mathcal{V}^{1-c-l}(\xi)\right)^{\Delta} = (1-c-l)\mathcal{V}^{\Delta}(\xi)\int_0^1 \left(h\mathcal{V}^{\sigma}(\xi)+(1-h)\mathcal{V}(\xi)\right)^{-c-l}dh$$

Since our assumptions imply that $\mathcal{V}^{\Delta}(\xi) = v(\xi) > 0$ and $0 \le c+l < 1$, we get

$$\left(\mathcal{V}^{1-c-l}(\xi)\right)^{\Delta} \le (1-c-l)\mathcal{V}^{\Delta}(\xi)\int_0^1 (h\mathcal{V}(\xi)+(1-h)\mathcal{V}(\xi))^{-c-l}dh.$$
$$= (1-c-l)v(\xi)\mathcal{V}^{-c-l}(\xi)$$

Thus

$$u(\xi) = \int_a^{\xi}\frac{v(\tau)}{\mathcal{V}^{c+l}(\tau)}\Delta\tau \ge \frac{1}{1-c-l}\int_a^{\xi}\left(\mathcal{V}^{1-c-l}(\tau)\right)^{\Delta}\Delta\tau$$

By employing (2.2) again to the term $\left(\mathcal{Z}^{l}(\xi)\right)^{\Delta}$, we see that

$$\left(\mathcal{Z}^{l}(\xi)\right)^{\Delta} = \left(\left(\int_0^{\xi} v(\tau)z(\tau)\Delta\tau\right)^{l}\right)^{\Delta}$$
$$= lv(\xi)z(\xi)\int_0^1\left(h\left(\int_0^{\sigma(\xi)} v(\tau)z(\tau)\Delta\tau\right)+(1-h)\left(\int_0^{\xi} v(\tau)z(\tau)\Delta\tau\right)\right)^{l-1}dh$$
$$\le lv(\xi)z(\xi)\left(\int_0^{\sigma(\xi)} v(\tau)z(\tau)\Delta\tau\right)^{l-1}$$

Thus

$$-\left(\mathcal{Z}^{l}(\xi)\right)^{\Delta} \ge -lv(\xi)z(\xi)\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l-1} \tag{3.27}$$

From (3.26) and (3.27), we get

$$-\int_a^b u(\xi)\left(\mathcal{Z}^{l}(\xi)\right)^{\Delta}\Delta\xi \ge \frac{l}{c+l-1}\int_a^b v(\xi)z(\xi)\mathcal{V}^{1-c-l}(\xi)\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l-1}\Delta\xi$$

Consequently

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l}\Delta\xi \ge \frac{l}{c+l-1}\int_a^b \frac{v(\xi)z(\xi)}{\mathcal{V}^{c+l-1}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l-1}\Delta\xi$$

By employing (2.6) on $\int_a^b \frac{v(\xi)z(\xi)}{\mathcal{V}^{c+l-1}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l-1}\Delta\xi$, with conjugate exponents $\frac{l}{l-1}$ and l, we obtain

$$\int_a^b \frac{v(\xi)z(\xi)}{\mathcal{V}^{c+l-1}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l-1}\Delta\xi \ge \left(\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l}\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_a^b \frac{v(\xi)z^{l}(\xi)}{\mathcal{V}^{c}(\xi)}\Delta\xi\right)^{\frac{1}{l}}$$

Consequently,

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l}\Delta\xi \ge \frac{l}{c+l-1}\left(\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l}\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_a^b \frac{v(\xi)z^{l}(\xi)}{\mathcal{V}^{c}(\xi)}\Delta\xi\right)^{\frac{1}{l}}.$$

Therefore,

$$\left(\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l}\Delta\xi\right)^{\frac{1}{l}} \ge \frac{l}{c+l-1}\left(\int_a^b \frac{v(\xi)z^{l}(\xi)}{\mathcal{V}^{c}(\xi)}\Delta\xi\right)^{\frac{1}{l}}$$

Thus,

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l}\Delta\xi \le \left(\frac{l}{c+l-1}\right)^{l}\int_a^b \frac{v(\xi)z^{l}(\xi)}{\mathcal{V}^{c}(\xi)}\Delta\xi$$

which provides (3.24).

Applying (2.5), with conjugate exponents $\frac{p}{p-l}$ and $\frac{p}{l}$, on $\int_a^b \frac{v(\xi)z^{l}(\xi)}{\mathcal{V}^{c}(\xi)}\Delta\xi$, we get

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\mathcal{Z}^{\sigma}(\xi)\right)^{l}\Delta\xi \le \left(\frac{l}{c+l-1}\right)^{l}\left(\int_a^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}}\left(\int_a^b \frac{v(\xi)}{\mathcal{V}^{cp}}(\xi)z^{p}(\xi)\Delta\xi\right)^{\frac{l}{p}}$$

which is (3.25).

Remark 3.9. If $\mathbb{T}=\mathbb{N}, b=\infty$ and $\{z_r\}_{r=1}^{\infty}, \{v_r\}_{r=1}^{\infty}$ are positive sequences in (3.24), then

$$\sum_{r=1}^{\infty} \frac{v_r}{\mathcal{V}_r^{c+l}}\left(\sum_{n=0}^{r} v_n z_n\right)^l \leq \left(\frac{l}{c+l-1}\right)^l \sum_{r=1}^{\infty} v_r z_r^l \mathcal{V}_r^{-c}$$

where,

$$\mathcal{V}_r=\sum_{n=1}^{r-1} v_n$$

Remark 3.10. If $\mathbb{T}=\mathbb{R}$ in (3.24), then we obtain the following inequality with a negative parameter

$$\int_a^b \frac{v(\xi)}{\mathcal{V}^{c+l}(\xi)}\left(\int_0^{\xi} v(\tau)z(\tau)\Delta\tau\right)^l d\xi \leq \left(\frac{l}{c+l-1}\right)^l \int_a^b \frac{v(\xi)z^l(\xi)}{\mathcal{V}^c(\xi)} d\xi$$

which is a corresponding formula to (1.13).

Theorem 3.5. Assume $a, b \in \mathbb{T}$ and $0<a<b\leq\infty$. If $\gamma-l\geq 0$, then

$$\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi \leq \left(\frac{-lK^{1+\gamma-l}}{1+\gamma-l}\right)^l \int_a^b \frac{v(\xi)z^l(\xi)}{\left(\mathcal{V}^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi \tag{3.28}$$

and

$$\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi \leq \left(\frac{-lK^{1+\gamma-l}}{1+\gamma-l}\right)^l \left(\int_a^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}} \left(\int_a^b \frac{v(\xi)z^p(\xi)}{\left(\mathcal{V}^\sigma(\xi)\right)^{p-\frac{\gamma p}{l}}}\Delta\xi\right)^{\frac{l}{p}} \tag{3.29}$$

where,

$$K=\inf_{\xi}\left(\frac{\mathcal{V}(\xi)}{\mathcal{V}^\sigma(\xi)}\right)^{1+\gamma-l}>0 \text{ and } \mathcal{Z}^*(\xi)=\int_0^{\xi}\frac{v(\tau)z(\tau)}{\mathcal{V}^\sigma(\tau)}\Delta\tau$$

Proof. By employing (2.3) to the right-hand side of (3.28) with

$$u^\Delta(\xi)=\frac{v(\xi)}{\left(\mathcal{V}^\sigma(\xi)\right)^{l-\gamma} \text{ and } w^\sigma(\xi)}=\left(\mathcal{Z}^*(\sigma(\xi))\right)^l$$

we get

$$\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi = u(\xi)\left(\mathcal{Z}^*(\xi)\right)^l\Big|_a^b - \int_a^b u(\xi)\left(\left(\mathcal{Z}^*(\xi)\right)^l\right)^\Delta \Delta\xi$$

where

$$u(\xi)=\int_a^{\xi}\frac{v(\tau)}{\left(\mathcal{V}^\sigma(\tau)\right)^{l-\gamma}}\Delta\tau$$

Since $u(a)=0$, we conclude that

$$\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi = u(b)\left(\mathcal{Z}^*(b)\right)^l - \int_a^b u(\xi)\left(\left(\mathcal{Z}^*(\xi)\right)^l\right)^\Delta \Delta\xi$$

$$\geq -\int_a^b u(\xi)\left(\left(\mathcal{Z}^*(\xi)\right)^l\right)^\Delta \Delta\xi$$

From (2.2), we see that

$$\left(\mathcal{V}^{1+\gamma-l}(\xi)\right)^\Delta=(1+\gamma-l)\mathcal{V}^\Delta(\xi)\int_0^1\left(h\mathcal{V}^\sigma(\xi)+(1-h)\mathcal{V}(\xi)\right)^{\gamma-l}dh$$

Since $\mathcal{V}^\Delta(\xi)=v(\xi)>0$ and $\gamma-l\geq 0$, we get

$$\left(\mathcal{V}^{1+\gamma-l}(\xi)\right)^\Delta \leq (1+\gamma-l)\frac{v(\xi)}{\left(\mathcal{V}^\sigma(\xi)\right)^{l-\gamma}}$$

Thus

$$\frac{v(\xi)}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}} \geq \frac{1}{1+\gamma-l}\left(\mathcal{V}^{1+\gamma-l}(\xi)\right)^{\Delta}$$

Consequently,

$$u(\xi) = \int_a^{\xi} \frac{v(\tau)}{\left(\mathcal{V}^{\sigma}(\tau)\right)^{l-\gamma}}\Delta\tau \geq \frac{1}{1+\gamma-l}\int_a^{\xi} \left(\mathcal{V}^{1+\gamma-l}(\tau)\right)^{\Delta}\Delta\tau$$

$$= \frac{1}{1+\gamma-l}\left(\frac{\mathcal{V}(\xi)}{\mathcal{V}^{\sigma}(\xi)}\right)^{1+\gamma-l}\left(\mathcal{V}^{\sigma}(\xi)\right)^{1+\gamma-l}$$

By employing (2.2) to the term $\left(\left(\mathcal{Z}^*(\xi)\right)^l\right)^{\Delta}$, we obtain

$$\left(\left(\mathcal{Z}^*(\xi)\right)^l\right)^{\Delta} = \left(\left(\int_0^{\xi} \frac{v(\tau)z(\tau)}{\mathcal{V}^{\sigma}(\tau)}\Delta\tau\right)^l\right)^{\Delta}$$

$$= l\left(\mathcal{Z}^*(\xi)\right)^{\Delta}\int_0^1 \left(h\mathcal{Z}^*(\sigma(\xi)) + (1-h)\mathcal{Z}^*(\xi)\right)^{l-1} dh$$

$$\leq l\left(\frac{v(\xi)z(\xi)}{\mathcal{V}^{\sigma}(\xi)}\right)\left(\mathcal{Z}^*(\sigma(\xi))\right)^{l-1}$$

Thus

$$-\left(\left(\mathcal{Z}^*(\xi)\right)^l\right)^{\Delta} \geq -l\left(\frac{v(\xi)z(\xi)}{\mathcal{V}^{\sigma}(\xi)}\right)\left(\mathcal{Z}^*(\sigma(\xi))\right)^{l-1} \tag{3.31}$$

From (3.30) and (3.31), we get

$$\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi \geq \frac{-lK^{1+\gamma-l}}{1+\gamma-l}\int_a^b \left(\frac{v(\xi)z(\xi)}{\mathcal{V}^{\sigma}(\xi)}\right)\left(\mathcal{V}^{\sigma}(\xi)\right)^{1+\gamma-l}\left(\mathcal{Z}^*(\sigma(\xi))\right)^{l-1}\Delta\xi$$

$$= \frac{-lK^{1+\gamma-l}}{1+\gamma-l}\int_a^b \left(\frac{v(\xi)z(\xi)}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\right)\left(\mathcal{Z}^*(\sigma(\xi))\right)^{l-1}\Delta\xi$$

By employing (2.6) on $\int_a^b \frac{v(\xi)z(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^{l-1}}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi$, with conjugate exponents $\frac{l}{l-1}$ and l, we see that

$$\int_a^b \frac{v(\xi)z(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^{l-1}}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi \geq \left(\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_a^b \frac{v(\xi)z^l(\xi)}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi\right)^{\frac{1}{l}}$$

Consequently,

$$\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi \geq \frac{-lK^{1+\gamma-l}}{1+\gamma-l}\left(\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_a^b \frac{v(\xi)z^l(\xi)}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi\right)^{\frac{1}{l}}$$

Therefore,

$$\left(\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi\right)^{\frac{1}{l}} \geq \frac{-lK^{1+\gamma-l}}{1+\gamma-l}\left(\int_a^b \frac{v(\xi)z^l(\xi)}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi\right)^{\frac{1}{l}}$$

Thus,

$$\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi \leq \left(\frac{-lK^{1+\gamma-l}}{1+\gamma-l}\right)^l\int_a^b \frac{v(\xi)z^l(\xi)}{\left(\mathcal{V}^{\sigma}(\xi)\right)^{l-\gamma}}\Delta\xi$$

which is (3.28).

By employing (2.5), with conjugate exponents $\frac{p}{p-l}$ and $\frac{p}{l}$, on $\int_a^b \frac{v(\xi)z^l(\xi)}{(\mathcal{V}^\sigma(\xi))^{l-\gamma}}\Delta\xi$, we get

$$\int_a^b \frac{v(\xi)\left(\mathcal{Z}^*(\sigma(\xi))\right)^l}{\left(\mathcal{V}^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi \leq \left(\frac{-lK^{1+\gamma-l}}{1+\gamma-l}\right)^l \left(\int_a^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}} \left(\int_a^b \frac{v(\xi)z^p(\xi)}{\left(\mathcal{V}^\sigma(\xi)\right)^{p-\frac{\gamma p}{l}}}\Delta\xi\right)^{\frac{l}{p}}$$

which is (3.29).

Remark 3.11. If $\gamma = l$ and $K = \inf_\xi \left(\frac{\mathcal{V}(\xi)}{\mathcal{V}^\sigma(\xi)}\right)$ in (3.28), then we obtain the following inequality with a negative parameter

$$\int_a^b v(\xi)\left(\int_0^\xi \frac{v(\tau)z(\tau)}{\mathcal{V}(\tau)}\Delta\tau\right)^l \Delta\xi \leq (-lK)^l \int_a^b v(\xi)z^l(\xi)\Delta\xi$$

which is a corresponding formula to the dynamic inequality proved by Saker in [37, Theorem 2.5].

Remark 3.12. If $\mathbb{T} = \mathbb{R}, b = \infty$ and $\gamma = l$ in (3.28), then

$$\int_a^\infty v(\xi)\left(\int_0^\xi \frac{v(\tau)z(\tau)}{\mathcal{V}(\tau)}d\tau\right)^l d\xi \leq (-lK)^l \int_a^\infty v(\xi)z^l(\xi)d\xi \tag{3.32}$$

If $v(\xi) = 1$ in (3.32), then we obtain the Copson-type inequality

$$\int_a^\infty \left(\int_0^\xi \frac{z(\tau)}{\tau - a}d\tau\right)^l d\xi \leq (-lK)^l \int_a^\infty z^l(\xi)d\xi$$

Remark 3.13. If $\mathbb{T} = \mathbb{N}$ and v_n, z_n are positive sequences in (3.28), then

$$\sum_{r=1}^\infty \frac{v_r}{\mathcal{V}_r^{l-\gamma}}\left(\sum_{n=0}^r \frac{v_n z_n}{\mathcal{V}_n}\right)^l \leq \left(\frac{lK^{1+\gamma-l}}{l-\gamma-1}\right)^l \sum_{r=1}^\infty \frac{v_r z_r^l}{\mathcal{V}_r^{l-\gamma}} \tag{3.33}$$

where $\mathcal{V}_r = \sum_{n=1}^r v_n$ and $\sum_{r=1}^\infty \frac{v_r z_r^l}{\mathcal{V}_r^{l-\gamma}} < \infty$.

If $\gamma = l$ in (3.33), then we obtain the discrete Copson-type inequality

$$\sum_{r=1}^\infty v_r\left(\sum_{n=0}^r \frac{v_n z_n}{\mathcal{V}_n}\right)^l \leq (-lK)^l \sum_{r=1}^\infty v_r z_r^l$$

Theorem 3.6. Assume $a, b \in \mathbb{T}$ and $b < \infty$. If $-1 < \gamma - l \leq 0$, then

$$\int_0^b \frac{v(\xi)\left(\overline{\mathcal{Z}}^*(\xi)\right)^l}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi \leq \left(\frac{-l}{1+\gamma-l}\right)^l \int_0^b \frac{v(\xi)z^l(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi \tag{3.34}$$

and

$$\int_0^b \frac{v(\xi)\left(\overline{\mathcal{Z}}^*(\xi)\right)^l}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi \leq \left(\frac{-l}{1+\gamma-l}\right)^l \left(\int_0^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}} \left(\int_0^b \frac{v(\xi)z^p(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{p-\frac{\gamma p}{l}}}\Delta\xi\right)^{\frac{l}{p}} \tag{3.35}$$

where,

$$\Lambda(\xi) = \int_\xi^b v(\tau)\Delta\tau \text{ and } \overline{\mathcal{Z}}^*(\xi) = \int_\xi^\infty \frac{v(\tau)z(\tau)}{\Lambda^\sigma(\tau)}\Delta\tau$$

Proof. By employing (2.4) to the right-hand side of (3.34) with

$$u^\Delta(\xi) = \frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma} \text{ and } w(\xi)} = \left(\overline{\mathcal{Z}}^*(\xi)\right)^l$$

we get

$$\int_0^b \frac{v(\xi)\left(\overline{Z}^*(\xi)\right)^l}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi = u(\xi)\left(\overline{Z}^*(\xi)\right)^l\Big|_a^b - \int_0^b u^\sigma(\xi)\left(\left(\overline{Z}^*(\xi)\right)^l\right)^\Delta \Delta\xi$$

where

$$u(\xi) = -\int_\xi^b \frac{v(\tau)}{\left(\Lambda^\sigma(\tau)\right)^{l-\gamma}}\Delta\tau$$

Since $u(b) = 0$, we conclude that

$$\int_0^b \frac{v(\xi)\left(\overline{Z}^*(\xi)\right)^l}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi = -u(a)\left(\overline{Z}^*(a)\right)^l - \int_0^b u^\sigma(\xi)\left(\left(\overline{Z}^*(\xi)\right)^l\right)^\Delta \Delta\xi$$

$$\geq -\int_0^b u^\sigma(\xi)\left(\left(\overline{Z}^*(\xi)\right)^l\right)^\Delta \Delta\xi$$

From (2.2), we see that

$$\left(\Lambda^{1+\gamma-l}(\xi)\right)^\Delta = (1+\gamma-l)\Lambda^\Delta(\xi)\int_0^1 \left(h\Lambda^\sigma(\xi) + (1-h)\Lambda(\xi)\right)^{\gamma-l} dh$$

Since our assumptions imply that $\Lambda^\Delta(\xi) = -v(\xi) < 0$ and $\gamma - l \leq 0$, we get

$$\left(\Lambda^{1+\gamma-l}(\xi)\right)^\Delta \geq -(1+\gamma-l)\frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}}$$

Thus

$$\frac{v(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}} \geq \frac{-1}{1+\gamma-l}\left(\Lambda^{1+\gamma-l}(\xi)\right)^\Delta.$$

Consequently,

$$-u^\sigma(\xi) = \int_{\sigma(\xi)}^b \frac{v(\tau)}{\left(\Lambda^\sigma(\tau)\right)^{l-\gamma}}\Delta\tau \geq \frac{-1}{1+\gamma-l}\int_{\sigma(\xi)}^b \left(\Lambda^{1+\gamma-l}(\tau)\right)^\Delta \Delta\tau$$

By employing (2.2) to the term $\left(\left(\overline{Z}^*(\xi)\right)^l\right)^\Delta$, we obtain

$$\left(\left(\overline{Z}^*(\xi)\right)^l\right)^\Delta = \left(\left(\int_\xi^\infty \frac{v(\tau)z(\tau)}{\Lambda^\sigma(\tau)}\Delta\tau\right)^l\right)^\Delta$$

$$= l\left(\overline{Z}^*(\xi)\right)^\Delta \int_0^1 \left(h\overline{Z}^*(\sigma(\xi)) + (1-h)\overline{Z}^*(\xi)\right)^{l-1} dh$$

From (3.36) and (3.37), we get

$$\int_0^b \frac{v(\xi)\left(\overline{Z}^*(\xi)\right)^l}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}}\Delta\xi \geq \frac{-l}{1+\gamma-l}\int_0^b \left(\frac{v(\xi)z(\xi)}{\Lambda^\sigma(\xi)}\right)\left(\Lambda^\sigma(\xi)\right)^{1+\gamma-l}\left(\overline{Z}^*(\xi)\right)^{l-1}\Delta\xi$$

$$= \frac{-l}{1+\gamma-l}\int_0^b \left(\frac{v(\xi)z(\xi)}{\left(\Lambda^\sigma(\xi)\right)^{l-\gamma}}\right)\left(\overline{Z}^*(\xi)\right)^{l-1}\Delta\xi$$

By employing (2.6) on $\int_0^b \left(\frac{v(\xi)z(\xi)}{(\Lambda^\sigma(\xi))^{l-\gamma}}\right)\left(\overline{Z}^*(\xi)\right)^{l-1}\Delta\xi$, with conjugate exponents $l/(l-1)$ and l, we see that

$$\int_0^b \frac{v(\xi)z(\xi)\left(\overline{Z}^*(\xi)\right)^{l-1}}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi \geq \left(\int_0^b \frac{v(\xi)\left(\overline{Z}^*(\xi)\right)^l}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_0^b \frac{v(\xi)z^l(\xi)}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi\right)^{\frac{1}{l}}$$

Consequently,

$$\int_0^b \frac{v(\xi)\left(\overline{\mathcal{Z}}^*(\xi)\right)^l}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi \geq \frac{-l}{1+\gamma-l}\left(\int_0^b \frac{v(\xi)\left(\overline{\mathcal{Z}}^*(\xi)\right)^l}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi\right)^{\frac{l-1}{l}}\left(\int_0^b \frac{v(\xi)z^l(\xi)}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi\right)^{\frac{1}{l}}$$

Therefore,

$$\left(\int_0^b \frac{v(\xi)\left(\overline{\mathcal{Z}}^*(\xi)\right)^l}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi\right)^{\frac{1}{l}} \geq \frac{-l}{1+\gamma-l}\left(\int_0^b \frac{v(\xi)z^l(\xi)}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi\right)^{\frac{1}{l}}$$

Thus

$$\int_0^b \frac{v(\xi)\left(\overline{\mathcal{Z}}^*(\xi)\right)^l}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi \leq \left(\frac{-l}{1+\gamma-l}\right)^l \int_0^b \frac{v(\xi)z^l(\xi)}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi$$

which is (3.34).

By employing (2.5), with conjugate exponents $p/(p-l)$ and p/l, on $\int_0^b \frac{v(\xi)z^l(\xi)}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi$, we get

$$\int_0^b \frac{v(\xi)\left(\overline{\mathcal{Z}}^*(\xi)\right)^l}{(\Lambda^\sigma(\xi))^{l-\gamma}}\Delta\xi \leq \left(\frac{-l}{1+\gamma-l}\right)^l\left(\int_0^b v(\xi)\Delta\xi\right)^{1-\frac{l}{p}}\left(\int_0^b \frac{v(\xi)z^p(\xi)}{(\Lambda^\sigma(\xi))^{p-\frac{\gamma p}{l}}}\Delta\xi\right)^{\frac{l}{p}}$$

which is (3.35).

Remark 3.14. If $\mathbb{T}=\mathbb{R}$ and $\gamma=l$ in (3.34), then

$$\int_0^b v(\xi)\left(\int_\xi^\infty \frac{v(\tau)z(\tau)}{\Lambda(\tau)}d\tau\right)^l d\xi \leq (-l)^l\int_0^b v(\xi)z^l(\xi)d\xi \quad (3.38)$$

If $v(\xi)=1$ in (3.38), then we obtain the Copson-type inequality.

$$\int_0^b\left(\int_\xi^\infty \frac{z(\tau)}{b-\tau}d\tau\right)^l d\xi \leq (-l)^l\int_0^b z^l(\xi)d\xi$$

Remark 3.15. If $\mathbb{T}=\mathbb{N}$ and v_n, z_n are positive sequences in (3.34), then we have that

$$\sum_{r=0}^{b-1}\frac{v_r}{\Lambda_r^{l-\gamma}}\left(\sum_{n=r}^{\infty}\frac{v_n z_n}{\Lambda_n}\right)^l \leq \left(\frac{l}{l-\gamma-1}\right)^l\sum_{r=0}^{b-1}\frac{v_r z_r^l}{\Lambda_r^{l-\gamma}} \quad (3.39)$$

where, $\Lambda_r=\sum_{n=r}^b v_n$ and $\sum_{r=0}^{b-1} v_r z_r^l/\Lambda_r^{l-\gamma}<\infty$.

Remark 3.16. If $\gamma=l$ in (3.39), then we get the discrete Copson-type inequality

$$\sum_{r=0}^{b-1} v_r\left(\sum_{n=r}^{\infty}\frac{v_n z_n}{\Lambda_n}\right)^l \leq (-l)^l\sum_{r=0}^{b-1} v_r z_r^l$$

4. Conclusion

In this paper, by making use of time scale calculus, we have derived some new Hardy-type weighted dynamic inequalities with negative exponents. The integral and discrete Hardy-type inequalities, presented as special cases of our main findings, are novel contributions. For future work, we plan to focus on deriving additional dynamic inequalities with negative exponents.

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